Technical Comments

Stress Compatibility Equations in Cylindrical Coordinates

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THE compatibility equations relating the components of strain in curvilinear coordinates are well-known.[1] Tuba, in a recent technical comment in this journal,[2] corrected Vlasov's[3] specialization of these equations to a plane orthogonal frame and their further reduction to the case of plane polar coordinates. His last equation [Eq. (5), Ref. 2] can readily be obtained from Brdička's earlier work[4] where these equations are given in cylindrical coordinates.

In stress analysis, especially when the surface tractions are specified, it is often more convenient to solve the stress rather than the displacement problem, for which it is necessary that the stress compatibility equations be satisfied. They are given here for reference purposes since a survey of the literature indicates that they are not available elsewhere;

$$(1+\nu)\nabla^2\tau_{xx} + \tau_{kk,xx} = -E\alpha\left(\frac{1+\nu}{1-\nu}\nabla^2 T + T_{,xx}\right) \times$$

$$(1+\nu)[\nabla^2\tau_{rr} - 4\tau_{r\theta,\theta}/r^2 - 2(\tau_{rr}-\tau_{\theta\theta})/r^2] + \tau_{kk,rr} = -E\alpha\left(\frac{1+\nu}{1-\nu}\nabla^2 T + T_{,rr}\right) \quad (1)$$

$$(1+\nu)[\nabla^2\tau_{\theta\theta} + 4\tau_{r\theta,\theta}/r^2 + 2(\tau_{rr}-\tau_{\theta\theta})/r^2] + \tau_{kk,r}/r + \tau_{kk,\theta\theta}/r^2 = -E\alpha\left(\frac{1+\nu}{1-\nu}\nabla^2 T + T_{,\theta\theta}/r^2 + T_{,r}/r\right)$$

$$(1+\nu)[\nabla^2\tau_{r\theta} + 2(\tau_{rr}-\tau_{\theta\theta})_{,\theta}/r^2 - 4\tau_{r\theta}/r^2] + (\tau_{kk,\theta}/r)_{,r} = -E\alpha(T_{,\theta}/r)_{,r}$$

$$(1+\nu)(\nabla^2\tau_{\theta x} + 2\tau_{rx,\theta}/r^2 - \tau_{\theta x}/r^2) + \tau_{kk,\theta x}/r = -E\alpha T_{,\theta x}/r$$

$$(1+\nu)(\nabla^2\tau_{rx} - 2\tau_{\theta x,\theta}/r^2 - \tau_{rx}/r^2) + \tau_{kk,rx} = -E\alpha T_{,rx}$$

where

$$\tau_{kk} = \tau_{xx} + \tau_{rr} + \tau_{\theta\theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

and where τ_{ij} are the physical components of stress, E is the modulus of elasticity, ν is Poisson's ratio, α is the coefficient of thermal expansion, and T is the temperature change from the quiescent state.

These equations reduce to those given by Lee[5] for the special case of both axial symmetry and temperatures varying only with the axial coordinate. In general, curvilinear coordinates these equations are

$$(1+\nu)\sigma_{ij|mn}g^{mn} + (\sigma_{mn}g^{mn})|_{ij} = -E\alpha\left(\frac{1+\nu}{1-\nu}T|_{mn}g^{mn}g_{ij} + T|_{ij}\right) \quad (2)$$

where the summation convention applies to a repeated index that represents covariance in one of the quantities in which it appears and contravariance in the other, and where i, j, m, n, can each take any of the values 1, 2, and 3; σ_{ij} are the covariant components of the second-rank stress tensor; $|_i$ is a covariant derivative with respect to x^i; g_{ij} are the covariant components of the metric tensor; and g^{ij} are the components of the contravariant reciprocal of the metric tensor.

Equation 2 reduces to Brdička's [Ref. 4, Eq. (54) and Ref. 6, Eq. (7)] when T is a linear function in an orthogonal Cartesian frame. It also reduces to that given by Boley and Wiener[7] for the case of an orthogonal Cartesian frame, and to Eq. (1) for the case of cylindrical coordinates.

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Comments on "Generalized Law of the Wall and Eddy-Viscosity Model for Wall Boundary Layers"

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IN a recent paper, Kleinstein[1] makes an analysis of the law of the wall for the hydraulically smooth case. Utilizing Prandtl's familiar mixing length theory in the form of an overlap layer viscosity function,

$$\epsilon^+ = k_1 u_\tau^+ y^+ \quad (1)$$

plus the cubic power sublayer variation theorized by Reichardt,[2]

$$\epsilon \sim y^3 \sim u^3 \quad (2)$$

Kleinstein derives a single formula for the law of the wall which is valid both in the sublayer and in the overlap layer:

$$y^+ = u^+ + (1/k_2)[\exp(k_1u^+) - 1 - k_1u^+ - \tfrac{1}{2}(k_1u^+)^2 - \tfrac{1}{6}(k_1u^+)^3] \quad (3)$$

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Kleinstein suggests $k_1 = 0.4$ and $k_2 = 7.7$ and compares Eq. (3) with the data of Lindgren,[3] which he regards as the most reliable measurements near the wall. As Kleinstein explains, Eq. (3) is accurate and algebraically simple and thus constitutes a great improvement over previous expressions by Deissler, van Driest, and Reichardt.

I should like to point out that Eq. (3), using exactly the same analysis and assumptions, was first derived by D. B. Spalding[4] in 1961. The equation has come to be known as "Spalding's formula" and has been used by many workers[5,8] in turbulent boundary-layer studies. Spalding himself suggested $k_2 = 9.025$, which is in slightly better agreement with both velocity and eddy-viscosity measurements than Kleinstein's proposed $k_2 = 7.7$. Incidentally, Spalding also noted the interesting fact that the earlier expressions of Deissler et al. did not satisfy Reichardt's cubic power condition.

Following the derivation of Eq. (3), which assumes constant shear stress across the layer, Kleinstein then makes an analysis of pipe flow, where the shear stress varies linearly, and shows that there is a Reynolds number effect both on the constants k_1 and k_2 and on the thickness of the sublayer. Such predicted Reynolds number effects undoubtedly must exist, at least in principle, but, as Mellor and Gibson[9,10] point out in their elegant studies, our knowledge of the eddy-viscosity is still too limited to make any precise predictions about shear stress effects. Moreover, Lindgren's data[3] does not show the slightest effect of Reynolds number. Ironically, the earlier data of Laufer,[11] which was the accepted standard at the time of Spalding's derivation, does indeed show the Reynolds number thinning of the sublayer as predicted by Kleinstein, although the effect is so slight that it could be fortuitous. In general, it is probably most fruitful at the present time simply to accept Eq. (3) as an excellent approximation for smooth wall flows, since it can then be used to estimate the convective acceleration in problems where the shear stress distribution is not known, as first pointed out by Brand and Persen.[8]

Finally, Kleinstein analyzes the problem of a flat plate with wall suction or injection, where the vertical velocity at the wall v_w is not zero. Using the approximate shear stress distribution,

$$(u_\tau^+)^2 = v_w^+ u^+ + 1 \tag{4}$$

which contains a misprint in Kleinstein's Eq. (35), he neatly derives a modified law of the wall for injection or suction which coincides with Stevenson's empirical deduction.[12] Agreement with experiment is fairly good and thus there is probably no need to improve the analysis, although Eq. (4) neglects the ordinary convective acceleration (when $v_w = 0$), the inclusion of which would place a significant negative term on the right hand side of Eq. (4), as shown by Brand and Persen.[8] However, in extending the analysis to predict the skin friction coefficient of the plate with suction or injection, the neglect of the additional convection term would cause a serious error.

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